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Hydrodynamic Limit of Coagulation-Fragmentation Type Models of *k*-Nary Interacting Particles

Vassili N. Kolokoltsov[1]

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Hydrodynamic limit of general k-nary mass exchange processes with discrete mass distribution is described by a system of kinetic equations that generalize classical Smoluchovski's coagulation equations and many other models that are intensively studied in the current mathematical and physical literature. Existence and uniqueness theorems for these equations are proved. At last, for k-nary mass exchange processes with $k > 2$ an alternative nondeterministic measure-valued limit (diffusion approximation) is discussed.

KEY WORDS: Interacting particles; k-nary interaction; measure-valued limits; kinetic equation; mass exchange processes; coagulation-fragmentation; diffusion approximation.

1. INTRODUCTION

1.1. Aims of the Paper

This paper is the third in the series of papers devoted to the Markov models of k-nary interacting particle systems and their measure-valued limits (see refs. 19 and 20). It deals with a special kind of interaction, which are intensively studied in the current mathematical and physical literature, namely to the coagulation-fragmentation models, and to more general mass exchange processes. The classical examples of these models are given by the Smoluchovski model of binary coagulation and its modifications which are characterized by various coagulation kernels and also by the possibility of the inverse process, i.e., fragmentation of a particle into a pair of smaller ones (see, e.g., refs. 1 and 30 for recent results and a bibliography on these

[1] School of Computing and Mathematics, Nottingham Trent University, Burton Street, Nottingham NG1 4BU, United Kingdom; e-mail: vassili.kolokoltsov@ntu.ac.uk

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models). In the present paper we shall extend these models to include not necessary binary coagulations (i.e., any number of particles can coagulate in one go), the fragmentation of particles to any number of pieces, and also more general processes, where, say, the rate of coagulation or fragmentation of two particles can be increased or decreased by the presence of a third particle, or where a particle can split another particle in pieces and coagulate with one of them. These and similar possibilities lead to a general kind of processes which could be called mass exchange processes. The aim of the paper is to show that as a number of particles go to infinity and under an appropriate (in fact, uniform) scaling of interaction rates, these processes converge to a measure-valued deterministic processes (hydrodynamic or mean field limits) described by a system of kinetic equations (system (1.7) below) that generalize Smoluchovski's equations (and its modifications that include possible fragmentation). We shall prove some existence and uniqueness results for these equations which constitute a far reaching generalization of the corresponding results obtained recently for the Smoluchovski equations. At the end of the paper we show that a different (nonuniform) scaling can lead to nondeterministic limits (e.g., of diffusion type) of our mass exchange processes.

Let us notice that the kinetic equations we obtain and analyze here represent a particular case of more general equations obtained formally (i.e., without any rigorous convergence or existence results) and by a different method in ref. 5. In fact, in ref. 5 we developed two such methods, one was suggested in ref. 4 and was based on the study of the evolution of the generating functionals and another was based on the idea of propagation of chaos (see, e.g., ref. 36).

It is worth mentioning some other related works on nonbinary interactions. Namely, in ref. 32 the coalescence with multiple collisions were studied and the corresponding models in which many of these multiple collisions can occur simultaneously were considered in refs. 9, 28, and 34. The fragmentation processes in which particles can break into any number of pieces were discussed in refs. 7 and 8.

When analyzing Smoluchovski's equations, it often occurred that the basic results were first obtained for a simplified model of discrete mass distribution and then were generalized to a more difficult models of continuous mass distribution. We shall adhere to this tradition considering here only discrete mass distribution and leaving the continuous models for the future work (see, e.g., ref. 21). Notice that for the simplified discrete models considered here, the measure-valued limits are described by processes on the spaces of sequences (measures on the set of natural numbers), but for more natural models with a continuous mass distribution, the same procedure will lead to processes with values in the spaces of Borel measures on $\mathbf{R}^+$ or on more general measurable spaces (see, e.g., ref. 30 for binary coagulations).

A further development of the theory should also include, of course, spatially nontrivial models, where the particles are characterized not only by their masses but also by their position in space (or other parameters), which is changing according to some given law, for example as a Brownian motion. In case of classical coagulation-fragmentation process, such spatially nontrivial models have been investigated recently in several papers, see, e.g., refs. 6, 11, 38, and references therein for discrete mass distribution and refs. 2 and 26 for continuous masses. Another important problem for the mass exchange processes considered here that should be addressed in the future is the estimate of the gelation times and the asymptotics of the large time behavior, see, e.g., refs. 10, 18, and 27 for this question in the context of the standard coagulation- fragmentation models.

1.2. Some Notations

We list here a few notations that will be used throughout the paper without further reminder:

$C_r^k = r(r-1)\cdots(r-k+1)/k!$ is a standard binomial coefficient defined for any real r and any positive integer k; in particular, it vanishes whenever $k>r$ and r is a positive integer; δ_i^j is the Kronecker symbol denoting 1 for $i=j$ and 0 otherwise;

$\mathbf{R}^\infty$ (respectively $\mathbf{R}_+^\infty$) is a linear space of all sequences $\{x_1, x_2,...\}$ of real numbers (respectively its subset with all x_j being nonnegative); $\mathbf{R}^\infty$ is considered to be a measurable space equipped with the usual σ-algebra of subsets generated by its finite-dimensional cylindrical subsets;

c_p, $p \geqslant 1$, denotes the Banach space of real sequences $x = \{x_1, x_2,...\}$ equipped with the norm $\|x\|_p = (\sum_{j=1}^\infty |x_j|^p)^{1/p}$;

c_∞ denotes the Banach spaces of real sequences with $\lim_{n\to\infty} x_n = 0$ equipped with the sup-norm $\|x\|_\infty = \sup_n |x_n|$;

$\mathbf{Z}^\infty$ (respectively $\mathbf{Z}_+^\infty$) is the set of sequences $N = \{n_1, n_2,...\}$ of integer numbers n_j (respectively its subset with all n_j being nonnegative) equipped with the usual partial order: $N \leqslant M = \{m_1, m_2,...\}$ means that $n_j \leqslant m_j$ for all j;

$\mathbf{R}_{+,\,\mathrm{fin}}^\infty$ and $\mathbf{Z}_{+,\,\mathrm{fin}}^\infty$ are the subsets of $\mathbf{R}_+^\infty$ and $\mathbf{Z}_+^\infty$ respectively with only finite number of nonvanishing coordinates; we shall denote by $\{e_j\}$ the standard basis in $\mathbf{R}_{+,\,\mathrm{fin}}^\infty$ and will occasionally represent the sequences $N = \{n_1, n_2,...\} \in \mathbf{Z}_{+,\,\mathrm{fin}}^\infty$ as the linear combinations $N = \sum_{j=1}^\infty n_j e_j$;

by $\mathscr{M}$ we shall denote the Banach space of real sequences $x = \{x_1, x_2, \ldots\}$ with the norm $\mu(x) = |x_1| + 2|x_2| + \cdots$ (letters $\mathscr{M}$ and μ come from the interpretation of $\mu(x)$ as the mass of the state x for $x \in \mathbf{R}^{\infty}_{+,\,\mathrm{fin}}$);

For a measurable space X, $B(X)$ denotes the Banach space of real bounded measurable functions on X equipped with the usual sup-norm; if X is a topological space, $C_b(X)$ denotes the Banach subspace of $B(X)$ consisting of continuous functions.

1.3. Discrete Mass Exchange Model and Its Hydrodynamic Limit

Suppose a particle is characterized by its mass m that can take only integer values. A collection of particles is then described by a vector $N = \{n_1, n_2, \ldots\} \in \mathbf{Z}^{\infty}_{+}$, where a nonnegative integer n_j denotes the number of particles of mass j. The state space of our model will be the set $\mathbf{Z}^{\infty}_{+,\,\mathrm{fin}}$ of finite collections of particles (i.e., of vectors N with only a finite number of positive n_j). We shall denote by $|N| = n_1 + n_2 + \cdots$ the number of particles in the state N, by $\mu(N) = n_1 + 2n_2 + \cdots$ the total mass of the particles in this state, and by $\mathrm{supp}(N) = \{j : n_j \neq 0\}$ the support of N considered as a measure on $\{1, 2, \ldots\}$. By a *mass exchange* we shall mean any transformation $\Psi \mapsto \Phi$ in $\mathbf{Z}^{\infty}_{+,\,\mathrm{fin}}$ such that $\mu(\Psi) = \mu(\Phi)$. For instance, if Ψ consists of only one particle, this transformation is *pure fragmentation*, and if Φ consists of only one particle, this transformation is *pure coagulation* (not necessarily binary, of course). By a process of *mass exchange with a given profile* $\Psi = \{\psi_1, \psi_2, \ldots\} \in \mathbf{Z}^{\infty}_{+,\,\mathrm{fin}}$ we shall mean the (conservative) Markov chain on $\mathbf{Z}^{\infty}_{+,\,\mathrm{fin}}$ specified by the Q-matrix Q^{Ψ} with the entries

$$Q^{\Psi}_{NM} = C^{\Psi}_{N} P^{M-N+\Psi}_{\Psi}, \qquad M \neq N, \tag{1.1}$$

where $C^{\Psi}_{N} = \prod_{i \in \mathrm{Supp}(\Psi)} C^{\psi_i}_{n_i}$ and $\{P^{\Phi}_{\Psi}\}$ is any collection of nonnegative numbers parametrized by $\Phi \in \mathbf{Z}^{\infty}_{+,\,\mathrm{fin}}$ such that $P^{\Phi}_{\Psi} = 0$ whenever $\mu(\Phi) \neq \mu(\Psi)$. Observe that since the mass is preserved, this Markov chain is effectively a chain with a finite state space (specified by the initial condition) and hence it is well defined by the matrix (1.1) and does not explode in finite time. Clearly, the behavior of the process defined by Q-matrix (1.1) is the following: (i) if $N \geqslant \Psi$ does not hold, then N is a stable state, (ii) if $N \geqslant \Psi$, then any randomly chosen subfamily Ψ of N, i.e., any ψ_1 particles of mass 1 from a given number n_1 of these particles, any ψ_2 particles of mass 2 from a given number n_2 etc (notice that the coefficient C^{Ψ}_{N} in (1.1) is just the number of these choices) can be transformed to a collection Φ with the rate P^{Φ}_{Ψ}.

Equivalently (and more appropriate for our purposes), the Markov chain with the Q-matrix (1.1) can be specified by a Markov semigroup on the space $B(\mathbf{Z}_{+,\,\mathrm{fin}}^{\infty})$ of bounded functions on $\mathbf{Z}_{+,\,\mathrm{fin}}^{\infty}$ with the generator

$$G_{\Psi} f(N) = C_N^{\Psi} \sum_{\Phi:\, \mu(\Phi)=\mu(\Psi)} P_{\Psi}^{\Phi}(f(N-\Psi+\Phi)-f(N)). \tag{1.2}$$

More generally, if k is a natural number, a *mass exchange process of order k* (or *k-nary mass exchange process*) is a (conservative) Markov chain on $\mathbf{Z}_{+,\,\mathrm{fin}}^{\infty}$ defined by the Q-matrix of the type $Q^k = \sum_{\Psi:\, |\Psi| \leqslant k} Q^{\Psi}$ with Q^{Ψ} given by (1.1) or equivalently by the generator $G_k = \sum_{\Psi:\, |\Psi| \leqslant k} G_{\Psi}$. More explicitly

$$G_k f(N) = \sum_{\Psi:\, |\Psi| \leqslant k,\, \Psi \leqslant N} C_N^{\Psi} \sum_{\Phi:\, \mu(\Phi)=\mu(\Psi)} P_{\Psi}^{\Phi}(f(N-\Psi+\Phi)-f(N)), \tag{1.3}$$

where P_{Ψ}^{Φ} is an arbitrary collection of nonnegative numbers that vanish whenever $\mu(\Psi) \neq \mu(\Phi)$. As in case of a single Ψ, for any initial state N, this Markov chain lives on a finite state space of all M with $\mu(M)=\mu(N)$ and hence is always well defined.

We shall now perform the following scaling. Choosing a positive real h, we shall consider instead of a Markov chain on $\mathbf{Z}_{+,\,\mathrm{fin}}^{\infty}$, a Markov chain on $h\mathbf{Z}_{+,\,\mathrm{fin}}^{\infty} \subset \mathbf{R}^{\infty}$ with the generator

$$(G_k^h f)(hN) = \frac{1}{h} \sum_{\Psi:\, |\Psi| \leqslant k,\, \Psi \leqslant N} h^{|\Psi|} C_N^{\Psi} \sum_{\Phi:\, \mu(\Phi)=\mu(\Psi)} P_{\Psi}^{\Phi}(f(Nh-\Psi h+\Phi h)-f(Nh)), \tag{1.4}$$

which can be considered as the restriction to $B(h\mathbf{Z}_{+,\,\mathrm{fin}}^{\infty})$ of an operator in $B(\mathbf{R}_{+,\,\mathrm{fin}}^{\infty})$ that we shall again denote by G_k^h and that acts as

$$(G_k^h f)(x) = \frac{1}{h} \sum_{\Psi:\, |\Psi| \leqslant k} C_{\Psi}^h(x) \sum_{\Phi:\, \mu(\Phi)=\mu(\Psi)} P_{\Psi}^{\Phi}(f(x-\Psi h+\Phi h)-f(x)), \tag{1.5}$$

where the function C_{Ψ}^h is defined as

$$C_{\Psi}^h(x) = \prod_{j \in \mathrm{Supp}(\Psi)} \frac{x_j(x_j-h)\cdots(x_j-(\psi_j-1)\,h)}{\psi_j!}$$

in case $x_j \geqslant (\psi_j-1)\,h$ for all j and $C_{\Psi}^h(x)$ vanishes otherwise. Clearly, as $h \to 0$, operator (1.5) converges formally (justification will be given in Section 4) to the operator Λ on $B(\mathbf{R}_{+,\,\mathrm{fin}}^{\infty})$ given by

$$\Lambda_k f(x) = \sum_{\Psi:\, |\Psi| \leqslant k} \frac{x^{\Psi}}{\Psi!} \sum_{\Phi:\, \mu(\Phi)=\mu(\Psi)} P_{\Psi}^{\Phi} \sum_{j=1}^{\infty} \frac{\partial f}{\partial x_j}(\phi_j-\psi_j), \tag{1.6}$$

where

$$x^{\Psi}=\prod_{j\in \mathrm{Supp}(\Psi)} x_j^{\psi_j}, \qquad \Psi!=\prod_{j\in \mathrm{Supp}(\Psi)} \psi_j!.$$

Operator (1.6) is an infinite dimensional first order partial differential operator, whose characteristics are described by the following infinite system of ordinary differential equations

$$\dot{x}_j=\sum_{\Psi:\,|\Psi|\leqslant k}\frac{x^{\Psi}}{\Psi!}\sum_{\Phi:\,\mu(\Phi)=\mu(\Psi)} P_{\Psi}^{\Phi}(\phi_j-\psi_j), \qquad j=1,2,\dots . \tag{1.7}$$

This is the general system of kinetic equations describing the hydrodynamic limit (in the terminology of ref. 29, say) of k-nary mass exchange processes with discrete mass distributions. In other contexts such a limit is also called mean-field or McKean–Vlasov limit (see, e.g., ref. 12 or ref. 36). System (1.7) is the main object for analysis in this paper.

1.4. Examples of Kinetic Equations (1.7)

In case of pure coagulation or fragmentation, $P_{\Psi}^{\Phi}\neq 0$ only if either $|\Psi|=1$ or $|\Phi|=1$. Denoting P_{Ψ}^{Φ} with $|\Phi|=1$ by Q_{Ψ} and P_{Ψ}^{Φ} with $|\Psi|=1$ by P^{Φ}, we can rewrite (1.6) as

$$\begin{aligned}(\Lambda_k f)(x)=&\sum_{\Phi:\,|\Phi|>1} x_{\mu(\Phi)}P^{\Phi}\left(\sum_{j=1}^{\mu(\Phi)-1}\frac{\partial f}{\partial x_j}\phi_j-\frac{\partial f}{\partial x_{\mu(\Phi)}}\right)\\ &+\sum_{\Psi:\,|\Psi|\leqslant k}\frac{x^{\Psi}}{\Psi!}Q_{\Psi}\left(\frac{\partial f}{\partial x_{\mu(\Psi)}}-\sum_{j=1}^{\mu(\Psi)-1}\psi_j\frac{\partial f}{\partial x_j}\right),\end{aligned} \tag{1.8}$$

and system (1.7) takes the form

$$\begin{aligned}\dot{x}_j=&\sum_{m=j+1}^{\infty} x_m\sum_{\Phi:\,\mu(\Phi)=m}P^{\Phi}\phi_j-x_j\sum_{\Phi:\,\mu(\Phi)=j}P^{\Phi}\\ &+\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)=j}Q_{\Psi}\frac{x^{\Psi}}{\Psi!}-\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)>j}Q_{\Psi}\frac{x^{\Psi}}{\Psi!}\psi_j.\end{aligned} \tag{1.9}$$

In particular, in the case of binary coagulation-fragmentation, i.e., if P^{Φ} and Q_{Ψ} do not vanish only for $|\Psi|=2$ and $|\Phi|=2$, one can write $P^{\Phi}=P^{ij}=P^{ji}$ for Φ consisting of two particles of mass i and j and similarly $Q_\Psi = Q_{ij} = Q_{ji}$ for Ψ consisting of only two particles of mass i and j. Hence (1.9) takes the form

$$\dot{x}_j = \sum_{m=j+1}^{\infty} x_m \tilde{P}^{m-j,j} - \frac{1}{2} x_j \sum_{l=1}^{j-1} \tilde{P}^{l,j-l} + \frac{1}{2} \sum_{i=1}^{j-1} Q_{i,j-i} x_i x_j - \sum_{m=j+1}^{\infty} Q_{m-j,j} x_{m-j} x_j, \tag{1.10}$$

where we introduced the notations $\tilde{P}^{ij} = P^{ij}$ for $i \neq j$ and $\tilde{P}^{ii} = 2P^{ii}$. System (1.10) is a usual system of equation describing the mean field limit of binary coagulation-fragmentation models (see, e.g., ref. 3), which turns to classical Smoluchovski's equation for a particular choice of coagulation-fragmentation kernels P^{ij} and Q_{ij}.

Another important particular case of (1.7) is obtained if one supposes that only binary interactions are allowed, i.e., if $P_\Psi^\Phi \neq 0$ only for $|\Psi| = 2$, which one can interpret as an assumption that any mass exchange can happen only as a result of a collision of two particles. Parametrizing profiles Ψ with $|\Psi| = 2$ by pairs (ij) (two particles of the mass i and j) we can then rewrite (1.7) as

$$\dot{x}_j = \frac{1}{2} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} x_k x_l \sum_{\Phi:\, \mu(\Phi) = l+k} P_{kl}^{\Phi} (\phi_j - \psi_j), \qquad j = 1, 2, \ldots . \tag{1.11}$$

A further natural restriction (or simplification) of the model is an assumption that any two particles can either coagulate forming one particle or exchange masses and be transformed again in two particles (i.e., fragmentation into three or more particles is not admissible). Then nonvanishing P_{kl}^Φ are either P_{kl}^{k+l} or P_{kl}^{ij} with $i + j = k + l$ and Eqs. (1.11) can be rewritten in the form

$$\dot{x}_j = \frac{1}{2} \sum_{i=j+1}^{\infty} \sum_{k=1}^{i-1} x_k x_{i-k} P_{k,i-k}^{j,i-j} + \frac{1}{2} \sum_{i=1}^{j-1} x_i x_{j-i} P_{j-i,i}^{j} - \sum_{k=1}^{\infty} x_k x_j \sum_{\Phi:\, \mu(\Phi) = k+j} P_{kj}^{\Phi}, \qquad j = 1, 2, \ldots \tag{1.12}$$

which is a well known system of coagulation equations with collision breakage, see, e.g., refs. 33 and 37 for physical discussion and ref. 27 for basic mathematical results. As noted in ref. 27, particular cases of (1.12) are given by (a discrete version of) the nonlinear breakage model studied in ref. CR and by a model of the evolution of raindrops size spectra discussed in ref. 35. Another particular case of (1.12) is a model when the masses i

and j of new particles formed after the collision of particles with masses k, l are some given functions of $\min(i, j)$. This model is considered in ref. 14 (Eq. (7) there) as an intermediate model connecting Smoluchovski's equations and a discrete mass version of the Oort–Hulst model in Safronov's form.[33] Let us notice at last that only slight modification in Markov model (1.5) and in given below mathematical proofs are needed to include a model with random injections of monomers from ref. 13 or the Oort–Hulst model discussed in refs. 14 and 25.

Finally let us observe that the main equation (1.7) can be written in the following equivalent form

$$\dot{x}_j = \sum_{l=1}^{k} \frac{1}{l!} \sum_{i_1=1}^{\infty} \cdots \sum_{i_l=1}^{\infty} x_{i_1} \cdots x_{i_l} \sum_{\Phi:\, \mu(\Phi) = i_1 + \cdots + i_l} P^{\Phi}_{i_1 e_1 + \cdots + i_l e_l} (\phi_j - \delta_j^{i_1} - \cdots - \delta_j^{i_l}), \tag{1.13}$$

which is sometimes more convenient to deal with.

1.5. Content of the Paper

In Section 2 we prove two results (Theorems 2.1 and 2.2) on the existence of the global solutions to system (1.7) subject to additional assumptions on the growth of the rates P^{Φ}_{Ψ}. Conservation of mass in these solutions is also discussed. In Section 3 we prove (Theorems 3.1 and 3.2) the uniqueness and continuous dependence on the initial data for the solutions of (1.7) under assumptions of Theorem 2.2 from Section 2. Then we discuss some consequences for the corresponding contraction semigroups on $B(\mathcal{M})$ (Theorem 3.3). The main result of the paper (Theorem 4.2) is given in Section 4, where we prove the convergence of the Markov process with generator (1.4) to a deterministic process in $\mathcal{M} \subset \mathbf{R}_+^\infty$ described by (1.7). In passing, we are giving here an alternative (probabilistic) proof of the main existence results of Section 2 (Theorem 4.1). Section 5 is devoted to a short discussion of the diffusion approximations to the discrete mass exchange processes which are available only for models with $k > 2$. Theorem 5.1 there is a consequence of the theory developed in refs. 20 and 23.

2. EXISTENCE OF SOLUTIONS FOR THE KINETIC EQUATIONS

For our mathematical study of kinetic equations we need some additional assumption that prevents the creation of a large number of equal (in particular, small) particles in one go. From now on, we shall assume that $\phi_j \leqslant k$ for all $\Phi = \{\phi_1,..., \phi_l\}$, $j = 1,..., l$, and Ψ such that $P^{\Phi}_{\Psi} \neq 0$ (the use of the same constant k for the bound of ϕ_j and the maximal order of interaction is surely not essential and is made only to reduce the number of constants).

This section is devoted to the problem of the existence of the global solutions to system (1.7) which we shall write in the vector form

$$\dot{x} = f(x), \qquad f(x) = \left\{ f_j(x) = \sum_{\Psi:\, |\Psi| \leqslant k} \frac{x^\Psi}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_\Psi^\Phi (\phi_j - \psi_j) \right\}. \tag{2.1}$$

We shall say that a function $x(t)$ on $[0, T)$ is a solution of (2.1) in the Banach space c_p or a *c_p-solution*, if $x(t) \in c_p$ for all $t \in [O, T)$, and moreover, the r.h.s. of (2.1) is well defined, and (2.1) holds, where $\dot{x}$ is defined with respect to c_p-norm. We say that $x(t)$ is a *solution of the integral version of* (2.1) with initial conditions $x(0)$, or a *weak solution*, if $x(t) = \{x_1(t), x_2(t),...\}$, where all x_j are continuous functions such that

$$x_j(t) = x_j(0) + \int_0^t f_j(x(\tau))\, d\tau$$

holds with the integrals being well-defined as Lebesgue integrals. We shall say that $x(t)$ is a *global solution*, if $T = \infty$. Clearly if $x(t)$ is a c_p-solution of (2.1), then it is also a c_q-solution for any $q \geqslant p$ and also a weak solution.

In the future, we shall often use the following sets of sequences with masses not exceeding or equal to a given positive number c:

$$\mathscr{M}_{\leqslant c} = \{x \in \mathscr{M} \cap \mathbf{R}_+^\infty : \mu(x) \leqslant c\}, \qquad \mathscr{M}_c = \{x \in \mathscr{M} \cap \mathbf{R}_+^\infty : \mu(x) = c\}.$$

By $\mathscr{P}_n$ we shall denote the natural finite-dimensional projections in $\mathbf{R}^\infty$ defined by

$$\mathscr{P}_n(\{x_1, x_2,...\}) = \{x_1,..., x_n, 0, 0,...\}.$$

Our first existence result is the following.

Theorem 2.1. (i) Suppose P_Ψ^Φ are such that

$$\sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_\Psi^\Phi = H(\Psi) \prod_{j=1}^\infty j^{\psi_j}, \tag{2.2}$$

where $H(\Psi)$ is a function of Ψ that tends to zero as $\mu(\Psi) \to \infty$. Then for any $x_0 \in c_\infty$ such that $\mu(x_0) < \infty$ there exists a global c_∞-solution $x(t)$ of (2.1) with the initial condition x_0 and such that $x(t) \in \mathscr{M}_{\leqslant \mu(x_0)}$ for all t.

(ii) Suppose P_Ψ^Φ are such that

$$\sum_{\Phi:\, \mu(\Phi)=\mu(\Psi)} P_\Psi^\Phi \leqslant C\left(\prod_{j=1}^{\infty} j^{\psi_j}\right)^{\alpha} \tag{2.3}$$

for all Ψ, where $C>0$ and $\alpha \in (0,1)$ are some constants. Then for any $p>1/(1-\alpha)$ and an arbitrary $x_0 \in c_p$ such that $\mu(x_0)<\infty$ there exists a global c_p-solution $x(t)$ of (2.1) with the initial condition x_0 and such that $x(t) \in \mathcal{M}_{\leqslant \mu(x_0)}$ for all t.

Remarks. (1) In case of binary pure coagulation, i.e., in case of Smoluchovski's equation (1.10) with vanishing $\tilde{P}^{ij}$, the estimate (2.2) reduces to the estimate $Q_{ij}=o(1)\, ij$ as $i, j \to \infty$, which is the best known condition that implies the existence of the global solution for this model, see, e.g., ref. 18. On the other hand, for Eq. (1.12) our estimate (2.2) reduces to the estimate under which the global existence of the solutions to (1.12) is proved in ref. 27. (2) The results on c_p-solutions in (i), (ii) may be new even for the classical equations (1.10), because usually one proves the existence of a weak solution (see, e.g., refs. 3, 18, 27, 29, and 30). (3) $\sum_\Phi P_\Psi^\Phi$ is the rate of decay of the profile Ψ and hence a natural quantity to impose an upper bound on it. Notice also that $\prod j^{\psi_j}$ in (2.2) is the product of masses of all particles in the profile Ψ.

The proof of Theorem 2.1 will be based on the two simple facts from calculus, which we formulate and prove here for completeness as Lemmas 2.1 and 2.2.

Lemma 2.1. Let B be a Banach space and K be its compact subset. Let f and f^n, $n=1,2,\dots$, be a uniformly bounded family of continuous functions $K \mapsto B$ such that $\lim_{n\to\infty} \|f^n(x)-f(x)\|=0$ in B uniformly for all $x \in K$. Moreover, suppose for any n and an $x_0^n \in K$ there exists a global solution $x^n(t)$ of equation $\dot{x}=f^n(x)$ in B with the initial condition $x^n(0)=x_0^n$ and such that $x^n(t) \in K$ for all t. Suppose the sequence x_0^n converges in B to some $x_0 \in K$. Then there exists a global solution of equation $\dot{x}=f(x)$ in B with the initial condition $x(0)=x_0$ and such that $x(t) \in K$ for all t.

Proof. As all x^n take values in a compact set, and the derivatives $\dot{x}^n(t)$ are uniformly bounded, one can choose a subsequence, of the sequence of functions $x^n(t)$, which we shall again denote by x^n, that converges to a function $x(t)$ uniformly for $t \leqslant T$ with an arbitrary T. Clearly $x(t)$ also takes values in K. Moreover, as

$$\|f^n(x^n(t))-f(x(t))\| \leqslant \|f^n(x^n(t))-f(x^n(t))\|+\|f(x^n(t))-f(x(t))\|,$$

and since f is continuous and hence uniformly continuous in K, we conclude that the sequence of derivatives $\dot{x}^n(t) = f^n(x^n(t))$ converges to $f(x(t))$ uniformly on $t \leqslant T$ for all T. Hence $\dot{x}(t)$ exists in B and equals $f(x(t))$.

Lemma 2.2. For any finite $c > 0$ the set $\mathscr{M}_{\leqslant c}$ is a compact subset of c_p (in the topology of c_p, of course, and not in the topology of $\mathscr{M}$) for any finite $p \geqslant 1$ or $p = \infty$.

Proof. (i) Let us prove that $\mathscr{M}_{\leqslant c}$ is closed. Suppose x^n, $n = 1, 2,...,$ is a sequence in $c_p \cap \mathscr{M}_{\leqslant c}$ and $x = \lim_{n \to \infty} x^n$ in c_p. Then $\mu(\mathscr{P}_l x^n) \leqslant \mu(x^n) \leqslant c$. As the convergence of a sequence x^n in each c_p implies the convergence of all finite-dimensional projections $\mathscr{P}_l x^n$, it follows that $\mu(\mathscr{P}_l x) \leqslant c$. This clearly implies $\mu(x) \leqslant c$, i.e., that $x \in \mathscr{M}_{\leqslant c}$.

(ii) Let us prove that $\mathscr{M}_{\leqslant c}$ is a pre-compact set. Let x^n be a sequence in $c_p \cap \mathscr{M}_{\leqslant c}$. By diagonal process one can choose a subsequence $x^{n'}$ and an element $x \in \mathbf{R}^\infty$ such that $\mathscr{P}_l x^{n'}$ converges to $\mathscr{P}_l x$ for all l. But such a convergence for a sequence from $\mathscr{M}_{\leqslant c}$ implies its convergence in any c_p with $p \geqslant 1$, because by choosing large enough l one can ensure that $x^n - \mathscr{P}_l x^n$ are uniformly small in c_p.

Proof of Theorem 2.1. (i) Let us first prove that $f(x)$ from (2.1) is uniformly bounded on any compact set $\mathscr{M}_{\leqslant c}$, $c < \infty$. Due to (2.2) and since $\phi_j \leqslant k$, $\psi_j \leqslant k$ for all j, Ψ, Φ,

$$\|f(x)\|_\infty \leqslant \sigma \sum_{\Psi:\, |\Psi| \leqslant k} \frac{x^\Psi}{\Psi!} \prod_{j=1}^{\infty} j^{\psi_j} = \sigma \sum_{l=1}^{k} \frac{1}{l!} (\mu(x))^l \leqslant \sigma \sum_{l=1}^{k} \frac{c^l}{l!}$$

on $\mathscr{M}_{\leqslant c}$, where $\sigma = 2k \sup_\Psi o(1)$.

Next, $f(x) \in c_\infty$ whenever $\mu(x) < \infty$. In fact, as $\phi_j \neq 0$ or $\psi_j \neq 0$ implies $\mu(\Psi) \geqslant j$, it follows that

$$|f_j(x)| \leqslant k \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \geqslant j} \frac{x^\Psi}{\Psi!} \prod_{i \in \operatorname{supp}(\Psi)} i^{\psi_i} o(1)_{\mu(\Psi) \to \infty} = o(1)_{j \to \infty} \sum_{l=1}^{k} \frac{1}{l!} (\mu(x))^l.$$

We shall now apply Lemma 2.1 in the Banach space c_∞ with the compact set $K = \mathscr{M}_{\leqslant c}$ using the finite-dimensional approximations f^n to f defined by

$$f_j^n(x) = \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant n} \frac{x^\Psi}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_\Psi^\Phi (\phi_j - \psi_j). \tag{2.4}$$

Clearly $f^n(x)=f^n(\mathscr{P}_n(x))=\mathscr{P}_n(f^n(x))$ for all x with a finite mass. Hence equations $\dot{x}=f^n(x)$ are finite-dimensional, i.e., $x(t)-\mathscr{P}_n(x(t))$ are constants along the solutions of these equations and $\mathscr{P}_n(x(t))$ satisfy the n-dimensional differential equations. Consequently, equations $\dot{x}=f^n(x)$ have unique solutions in $\mathscr{M}_{\leqslant c}$ for any $c<\infty$ and $x_0\in\mathscr{M}_{\leqslant c}$ (notice that the vector field $f^n(x)$ is nowhere pointing outside $\mathscr{M}_{\leqslant c}$ on its border and hence a solution is forced to stay in $\mathscr{M}_{\leqslant c}$ whenever $x_0\in\mathscr{M}_{\leqslant c}$). As all f^n are uniformly bounded on each $\mathscr{M}_{\leqslant c}$ (the same proof as for f above), to deduce the statement (i) from Lemma 2.1 it remains to show that $\|f^n(x)-f(x)\|_\infty\to 0$ as $n\to\infty$ uniformly for $x\in\mathscr{M}_{\leqslant c}$. This is true because of the estimate

$$\begin{aligned}\|f^n(x)-f(x)\|_\infty &\leqslant k\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)\geqslant n}\frac{x^\Psi}{\Psi!}\prod_{j=1}^{\infty} j^{\psi_j}o(1)_{n\to\infty}\\ &=o(1)_{n\to\infty}\sum_{l=1}^{k}\frac{1}{l!}(\mu(x))^l.\end{aligned}$$

(ii) We shall follow the same line of argument as in (i) and will use the same approximation (2.4) and Lemma 2.1 in the Banach spaces c_p with $p>1/(1-\alpha)$. First let us show that under (2.3), f and f^n are uniformly bounded in c_p. As $|\Psi|\leqslant k$, it follows that if $\phi_j\neq 0$ or $\psi_j\neq 0$, then $\mu(\Psi)=\mu(\Phi)\geqslant j$ and hence there exists $i\geqslant j/k$ such that $\psi_i\neq 0$. Hence

$$\begin{aligned}|f_j(x)| &\leqslant 2kC\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)\geqslant j}\frac{x^\Psi}{\Psi!}\prod_{i\in\operatorname{supp}(\Psi)}(i^{\psi_i})^\alpha\\ &\leqslant 2kC\sum_{\Psi:\,|\Psi|\leqslant k,\,\exists l\geqslant j/k:\,\psi_l>0}\frac{x^\Psi}{\Psi!}\prod_{i\in\operatorname{supp}(\Psi)}(i^{\psi_i})^\alpha.\end{aligned}$$

Since for any Ψ such that $\psi_l\neq 0$ for some $l\geqslant j/k$,

$$\prod_{i\in\operatorname{supp}(\Psi)}(i^{\psi_i})^\alpha=(l^{\psi_l})^{-(1-\alpha)}\,l^{\psi_l}\prod_{i\neq l}(i^{\psi_i})^\alpha\leqslant k^{1-\alpha}j^{-(1-\alpha)}\prod_{i\in\operatorname{supp}(\Psi)}i^{\psi_i},$$

we conclude that

$$\begin{aligned}|f_j(x)| &\leqslant 2Ck^{2-\alpha}j^{-(1-\alpha)}\sum_{\Psi:\,|\Psi|\leqslant k}\frac{x^\Psi}{\Psi!}\prod_{i\in\operatorname{supp}(\Psi)}i^{\psi_i}\\ &=2Ck^{2-\alpha}j^{-(1-\alpha)}\sum_{l=1}^{k}\frac{1}{l!}(\mu(x))^l,\end{aligned}\tag{2.5}$$

and hence $\|f(x)\|_p$ is uniformly bounded in $\mathscr{M}_{\leqslant c}$ for any $p > 1/(1-\alpha)$ and $c < \infty$. Similarly all $\|f^n\|_p$ are bounded. To deduce statement (ii) from Lemma 2.1 it remains to show that $\|f^n(x) - f(x)\|_p \to 0$ as $n \to \infty$. To this end we estimate

$$
\begin{aligned}
|f_j^n(x) - f_j(x)| &\leqslant 2kC \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \geqslant \max(n,\, j)} \frac{x^\Psi}{\Psi!} \prod_{i \in \operatorname{supp}(\Psi)} (i^{\psi_i})^\alpha \\
&\leqslant 2Ck^{2-\alpha}[\max(n, j)]^{-(1-\alpha)} \sum_{l=1}^{k} \frac{1}{l!} (\mu(x))^l,
\end{aligned}
$$

and hence

$$
\|f^n(x) - f(x)\|_p \leqslant 2Ck^{2-\alpha} \left[n^{-(p(1-\alpha)-1)} + \sum_{j>n} j^{-p(1-\alpha)} \right]^{1/p} \sum_{l=1}^{k} \frac{1}{l!} (\mu(x))^l, \tag{2.6}
$$

which tends to zero as $n \to \infty$ for $p > 1/(1-\alpha)$. Proof of Theorem 2.1 is complete.

We shall discuss now the existence of the mass preserving solutions by a generalization of a (rather standard by now) method of higher mass moments in the spirit of papers of refs. 27 and 29. For brevity, we shall use only the second mass moments (generalization to other moments of order $j > 1$ with the corresponding modifications and improvements of final results are more or less straightforward). The second moment for a $x \in \mathbf{R}_+^\infty$ is defined as

$$
\mu_2(x) = \sum_{j=1}^{\infty} j^2 x_j. \tag{2.7}
$$

The corner stone in the proof of the next theorem is given by the following estimate on the evolution of the second mass moments.

Lemma 2.3. Suppose

$$
\sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_\Psi^\Phi \leqslant C\mu(\Psi) \tag{2.8}
$$

for some constant C. Let $x(t)$ be a solution of the finite-dimensional system $\dot{x} = f^n(x)$ with f_n given by (2.4) and with the initial point $x_0 \in \mathbf{R}_+^n$. Then

$$
\mu_2(x(t)) \leqslant e^{at}(\mu_2(x_0) + b) \tag{2.9}
$$

with constants a, b depending only on C from (2.8), k and $\mu(x_0)$.

Proof. For brevity, we shall write simply x for $x(t)$. From (2.4) it follows that

$$\frac{d}{dt}\mu_2(x) = \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant n} \frac{x^\Psi}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_\Psi^\Phi \sum_j j^2(\phi_j - \psi_j).$$

The first key observation (which is easily seen by inspection) is that

$$\max_{\Phi:\, \mu(\Phi) = \mu(\Psi)} \mu_2(\Phi) = (\mu(\Psi))^2, \tag{2.10}$$

which together with (2.8) implies that

$$\frac{d}{dt}\mu_2(x) \leqslant C \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant n} \frac{x^\Psi}{\Psi!} \mu(\Psi)[(\mu(\Psi))^2 - \mu_2(\Psi)]. \tag{2.11}$$

To make another step, let us use the multinomial formula to rewrite this estimate in the following equivalent but more explicit form

$$\frac{d}{dt}\mu_2(x)$$
$$\leqslant C \sum_{i=1}^k \frac{1}{i!} \sum_{j_1=1}^n \cdots \sum_{j_i=1}^n x_{j_1} \cdots x_{j_i} (j_1 + \cdots + j_i)[(j_1 + \cdots + j_i)^2 - (j_1^2 + \cdots + j_i^2)]$$

and then use the explicit symmetry of indexes $j_1, \ldots, j_i$ to again rewrite it as

$$\frac{d}{dt}\mu_2(x)$$
$$\leqslant C \sum_{i=2}^k \frac{1}{(i-1)!} \sum_{j_1=1}^n \cdots \sum_{j_i=1}^n x_{j_1} \cdots x_{j_i} j_1 [(j_1 + \cdots + j_i)^2 - (j_1^2 + \cdots + j_i^2)]. \tag{2.12}$$

The next step is now to use the obvious equality

$$(j_1 + \cdots + j_i)^2 - (j_1^2 + \cdots + j_i^2) = j_1 j_2 + \cdots + j_1 j_i + j_2 j_3 + \cdots + j_2 j_i + \cdots$$

and the symmetry of indexes $j_2, \ldots, j_i$ in (2.12) to rewrite (2.12) as

$$\frac{d}{dt}\mu_2(x) \leqslant C \sum_{i=2}^k \frac{1}{(i-2)!} \sum_{j_1=1}^n \cdots \sum_{j_i=1}^n x_{j_1} \cdots x_{j_i} j_1^2 j_2$$
$$+ \frac{1}{2} C \sum_{i=3}^k \frac{1}{(i-3)!} \sum_{j_1=1}^n \cdots \sum_{j_i=1}^n x_{j_1} \cdots x_{j_i} j_1 j_2 j_3.$$

Increasing the r.h.s. of this inequality one obtains

$$\begin{aligned}\frac{d}{dt}\mu_2(x) &\leqslant C\sum_{i=2}^{k}\frac{1}{(i-2)!}\sum_{j_1=1}^{n}\cdots\sum_{j_i=1}^{n}x_{j_1}\cdots x_{j_i}j_1^2 j_2\cdots j_i\\ &\quad+\frac{1}{2}C\sum_{i=3}^{k}\frac{1}{(i-3)!}\sum_{j_1=1}^{n}\cdots\sum_{j_i=1}^{n}x_{j_1}\cdots x_{j_i}j_1 j_2\cdots j_i\\ &=C\sum_{i=2}^{k}\frac{1}{(i-2)!}\mu_2(x)(\mu(x))^{i-1}+\frac{C}{2}\sum_{i=3}^{k}\frac{1}{(i-3)!}(\mu(x))^i.\end{aligned}$$

Hence

$$\frac{d}{dt}\mu_2(x)\leqslant a\mu_2(x)+\beta \tag{2.13}$$

with

$$a=C\sum_{i=2}^{k}\frac{1}{(i-2)!}(\mu(x_0))^{i-1},\qquad \beta=\frac{C}{2}\sum_{i=3}^{k}\frac{1}{(i-3)!}(\mu(x_0))^i.$$

Clearly (2.13) implies (2.9) with $b=a/\beta$ (say, by Gronwall's lemma), and Lemma 2.3 is proved.

We can now prove our second result on the existence of solutions to (1.7). To this end, let us denote by $\mathcal{M}_c^2$ (for any positive finite c) the set of all sequences from $\mathcal{M}_c$ with a finite second mass moment, i.e.

$$\mathcal{M}_c^2=\{x\in\mathcal{M}\cap\mathbf{R}_+^\infty:\mu(x)=c,\ \mu_2(x)<\infty\}.$$

Theorem 2.2. Suppose (2.8) holds and $x\in\mathcal{M}_c^2$ with some $c>0$. Then for all $p>1$ there exists a global c_p-solution of (2.1) with the initial condition x_0 such that $x(t)\in\mathcal{M}_c^2$ for all t; in particular, the conservation of mass equation holds, i.e.

$$\mu(x(t))=\mu(x_0). \tag{2.14}$$

Remarks. (1) Notice that (2.8) does not imply (2.2) and hence even the existence of a solution does not follow directly from Theorem 2.1. (2) Seemingly it is possible to prove the existence of a mass conserving solution without the assumption that $\mu_2(x)<\infty$ by generalizing the corresponding arguments from ref. 27 or ref. 3. We choose here more restrictive assumptions which, on the one hand, require a much shorter proof, and on the other hand, coincide with the assumptions that we need to prove uniqueness in the next section. (3) In the case of binary coagulation-fragmentation models, our conditions (2.2) and (2.8) coincide with the usually used growth conditions, see, e.g., ref. 30.

Proof. We shall prove the existence of the solutions on $t \in [0, T]$ with an arbitrary fixed T by again using Lemma 2.1 in the Banach space c_p with any $p > 1$, with the same approximations f^n from (2.4), and with the compact set

$$K_T = \{x \in c_p : \mu(x) \leqslant \mu(x_0),\ \mu_2(x) \leqslant e^{aT}\mu_2(x_0) + b\}$$

(a proof that K_T is a compact set is done by precisely the same arguments as in the proof of Lemma 2.2 above).

Let $x_0^n = \mathscr{P}_n x_0$. Then there exists a unique solution $x^n(t)$ of $\dot{x} = f^n(x)$ in $\mathbf{R}_+^n$ with the initial condition x_0^n. By Lemma 2.3, $x^n(t) \in K_T$ for all n and $t \leqslant T$. As clearly

$$\mu(\Psi) \leqslant |\Psi| \prod_{i \in \operatorname{supp}(\Psi)} i^{\psi_i} \tag{2.15}$$

for any profile $\Psi \in \mathbf{Z}_{+,\,\mathrm{fin}}^{\infty}$, (2.8) implies (2.3) with $\alpha = 1$ and we get

$$|f_j^n(x)| \leqslant Ck^2 \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \geqslant j} \frac{x^\Psi}{\Psi!} \prod_{i \in \operatorname{supp}(\Psi)} i^{\psi_i}$$

for any x and then by the same argument as used when proving (2.5) we estimate the r.h.s. of this inequality by

$$2Ck^3 \sum_{\Psi:\, |\Psi| \leqslant k} \frac{x^\Psi}{\Psi!} \frac{1}{j} \prod_{i \in \operatorname{supp}(\Psi)} i^{2\psi_i} = 2Ck^3 \frac{1}{j} \sum_{l=1}^{k} \frac{1}{l!} (\mu_2(x))^l.$$

Hence for any $p > 1$, the norms $\|f^n(x)\|_p$ are uniformly bounded for $x \in K_T$. Let us prove that $\|f^n(x) - f(x)\|_p$ tends to zero as $n \to \infty$ uniformly for all $x \in K_T$. As in the proof of estimate (2.6) we get

$$\begin{aligned} |f_j^n(x) - f_j(x)| &\leqslant 2k^2C \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \geqslant \max(n,\, j)} \frac{x^\Psi}{\Psi!} \prod_{i \in \operatorname{supp}(\Psi)} i^{\psi_i} \\ &\leqslant 2Ck^3 [\max(n, j)]^{-1} \sum_{l=1}^{k} \frac{1}{l!} (\mu_2(x))^l, \end{aligned}$$

and hence

$$\|f^n(x) - f(x)\|_p \leqslant 4Ck^3 \left[n^{-(p-1)} + \sum_{j>n} j^{-p} \right]^{1/p} \sum_{l=1}^{k} \frac{1}{l!} (\mu_2(x))^l,$$

which tends to zero as $n \to \infty$ for $p > 1$. As T is arbitrary, we get the existence of a global solution by Lemma 2.1. As K_T is a compact set, the obtained solution belongs to K_T on $[0, T]$ and consequently it always has the finite mass moment $\mu_2(x(t))$.

It remains to show (2.14). This is simple. In fact, as the conservation of mass holds for the approximations x^n, it is enough to prove that

$$\lim_{n \to \infty} |\mu(x^n(t)) - \mu(x(t))| = 0.$$

This is true, because on the one hand,

$$\lim_{n \to \infty} |\mu(\mathscr{P}_l x^n(t)) - \mu(\mathscr{P}_l x(t))| = 0$$

for any l, and on the other hand, both $\mu(x_n(t) - \mathscr{P}_l x^n(t))$ and $\mu(x(t) - \mathscr{P}_l x(t))$ can be made uniformly (in n) arbitrary small by choosing large enough l due to the uniform bound on the second mass moment. Theorem 2.2 is thus proved.

We shall conclude this section by proving a lower bound for the growth of $\mu_2(x)$ on the solutions of $\dot{x} = f^n(x)$ (similar to the upper bound (2.13)) that we shall use in the next section.

Lemma 2.4. Under assumptions of Lemma 2.3 suppose additionally that there exists a constant $\omega \geqslant 0$ such that either

$$\mu_2(\Phi) - \mu_2(\Psi) \geqslant -\omega \mu(\Psi) \tag{2.16}$$

whenever $P_\Psi^\Phi \neq 0$, or that for all Ψ

$$\sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_\Psi^\Phi \leqslant \omega. \tag{2.17}$$

Then for any solution $x(t)$ of the equation $\dot{x} = f^n(x)$ one has

$$\frac{d}{dt} \mu_2(x(t)) \geqslant -\tilde{a} \mu_2(x(t)) - \tilde{\beta} \tag{2.18}$$

and hence

$$\mu_2(x(t)) \geqslant e^{-\tilde{a}t} \mu_2(x_0) - \tilde{b} \tag{2.19}$$

with some nonnegative constants $\tilde{a}$, $\tilde{b}$, $\tilde{\beta}$.

Remark. Of course, condition (2.16) is restrictive. However, it holds in many important situations. For example, it holds for processes of pure coagulation with $\omega=0$. It holds for Becker–Döring equations (see ref. 3 and references therein) and for the generalized Becker–Döring models introduced in ref. 18. Roughly speaking, condition (2.16) forbids fragmentation into very small pieces in one go. A discussion of the applicability of condition (2.17), is given in ref. 2.

Proof. It is similar to the proof of Lemma 2.3. Using (2.16) instead of (2.10), or (2.17), yields instead of (2.11) the estimate

$$\frac{d}{dt}\mu_2(x) \geqslant -\tilde{C} \sum_{\Psi:\,|\Psi|\leqslant k,\, \mu(\Psi)\leqslant n} \frac{x^{\Psi}}{\Psi!}(\mu(\Psi))^2,$$

where $\tilde{C}=C\omega$ in case (2.16) or $\tilde{C}=\omega$ in case (2.17). Consequently

$$\frac{d}{dt}\mu_2(x)$$

$$\geqslant -\tilde{C}\sum_{i=1}^{k}\frac{1}{i!}\sum_{j_1=1}^{n}\cdots\sum_{j_i=1}^{n} x_{j_1}\cdots x_{j_i}[j_1^2+\cdots+j_i^2+2j_1j_2+2j_1j_3+\cdots+j_2j_3+\cdots],$$

and by symmetry

$$\begin{aligned}\frac{d}{dt}\mu_2(x) &\geqslant -\tilde{C}\sum_{i=1}^{k}\frac{1}{(i-1)!}\sum_{j_1=1}^{n}\cdots\sum_{j_i=1}^{n} x_{j_1}\cdots x_{j_i}j_1^2\\ &\quad -\frac{1}{2}\tilde{C}\sum_{i=2}^{k}\frac{1}{(i-2)!}\sum_{j_1=1}^{n}\cdots\sum_{j_i=1}^{n} x_{j_1}\cdots x_{j_i}j_1j_2\\ &\geqslant -\tilde{C}\sum_{i=1}^{k}\frac{1}{(i-1)!}\mu_2(x)(\mu(x))^{i-1}-\frac{\tilde{C}}{2}\sum_{i=2}^{k}\frac{1}{(i-2)!}(\mu(x))^i,\end{aligned}$$

which is precisely (2.18). Clearly (2.19) is a consequence of (2.18).

3. UNIQUENESS AND CONTINUOUS DEPENDENCE ON INITIAL DATA

The main objective of this section is to prove the following result.

Theorem 3.1. Suppose (2.8) holds. Let $\xi=\{\xi_1,\xi_2,...\}\in\mathcal{M}_c^2$, $\eta=\{\eta_1,\eta_2,...\}\in\mathcal{M}_{\tilde{c}}^2$ with some positive c and $\tilde{c}$. Let $x(t)$, $y(t)$ be any global weak or c_∞-solutions of (2.1) with the initial conditions ξ and η respectively and such that $\mu_2(x(t))$ and $\mu_2(y(t))$ are uniformly bounded on $t\in[0,T]$

for any $T > 0$ (the existence of these solutions is ensured by Theorem 2.2). Then for all t

$$\mu(x(t) - y(t)) \leqslant C(T)\, \mu(\xi - \eta), \tag{3.1}$$

where $C(T)$ is a constant depending on T, and on c, k and the bounds for $\mu_2(x(t))$ and $\mu_2(y(t))$ on $[0, T]$.

Remark. Recall that $\mu(\xi - \eta) = \sum_{i=1}^{\infty} i\, |\xi_i - \eta_i|$; this function clearly defines a metric on $\mathscr{M}_c^2$ and $\mathscr{M}_c$ with respect to which both these spaces are complete metric spaces.

The main idea of the proof of Theorem 3.1 is the same as used in refs. 3 and 27 for Eqs. (1.10) and (1.12), respectively. Two basic technical ingredients of this proof are obtained below as Lemmas 3.1 and 3.2. We shall use the obvious observation that the uniform boundedness of $\mu_2(x(t))$ implies the uniform boundedness of $\mu(x(t))$.

Lemma 3.1. Suppose (2.8) holds. Let $x(t)$ be a solution of (2.1) that satisfies the assumptions of Theorem 3.1. Then for all T

$$\lim_{n \to \infty} \int_0^T \sum_{i=1}^{n} i \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \geqslant n} \frac{(x(t))^{\Psi}}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_{\Psi}^{\Phi} \psi_i \, dt = 0, \tag{3.2}$$

$$\lim_{n \to \infty} \int_0^T \sum_{i=1}^{n} i \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \geqslant n} \frac{(x(t))^{\Psi}}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_{\Psi}^{\Phi} \phi_i \, dt = 0. \tag{3.3}$$

Proof. First let us show that (3.3) is a consequence of (3.2). As

$$\sum_{i=1}^{n} i x_i(t) = \sum_{i=1}^{n} i \xi_i + \sum_{i=1}^{n} i \int_0^T f_i(x(t))\, dt,$$

(2.1) and (2.13) imply that

$$\lim_{n \to \infty} \int_0^T \sum_{i=1}^{n} i \sum_{\Psi:\, |\Psi| \leqslant k} \frac{(x(t))^{\Psi}}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_{\Psi}^{\Phi} (\phi_i - \psi_i)\, dt = 0. \tag{3.4}$$

But

$$\begin{aligned} &\sum_{i=1}^{n} i \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant n} \frac{(x(t))^{\Psi}}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_{\Psi}^{\Phi} (\phi_i - \psi_i) \\ &\quad = \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant n} \frac{(x(t))^{\Psi}}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_{\Psi}^{\Phi} (\mu(\Phi) - \mu(\Psi)) = 0. \end{aligned}$$

Consequently, (3.4) implies

$$\lim_{n\to\infty}\int_0^T\sum_{i=1}^n i\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)\geqslant n}\frac{(x(t))^\Psi}{\Psi!}\sum_{\Phi:\,\mu(\Phi)=\mu(\Psi)}P_\Psi^\Phi(\phi_i-\psi_i)\,dt=0,$$

and hence (3.2) and (3.3) are equivalent.

Now let us prove (3.2). By (2.8) and the uniform boundedness of $\mu_2(x(t))$, it is enough to show that

$$\lim_{n\to\infty}\sum_{i=1}^n i\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)\geqslant n}\frac{x^\Psi}{\Psi!}\mu(\Psi)\,\psi_i=0 \tag{3.5}$$

uniformly for all x with uniformly bounded $\mu_2(x)$ (and hence also $\mu(x)$). As $\psi_i\leqslant k$ and as x_i are supposed to be uniformly bounded, to prove (3.5) it is sufficient to show that

$$\lim_{n\to\infty}\sum_{i=1}^n ix_i\sum_{\Psi:\,|\Psi|\leqslant k-1,\,\mu(\Psi)\geqslant n-ki,\,\psi_i=0}\frac{x^\Psi}{\Psi!}(i+\mu(\Psi))=0. \tag{3.6}$$

We shall represent the l.h.s. of (3.6) as the sum of two terms by writing the sum $\sum_{i=1}^n$ as the the sum of two sums over $i\geqslant n/(2k)$ and over $i<n/(2k)$ respectively. To prove that the first term tends to zero it is enough to show that

$$\lim_{n\to\infty}\sum_{i=n}^\infty ix_i\sum_{\Psi:\,|\Psi|\leqslant k,\,\psi_i=0}\frac{x^\Psi}{\Psi!}(i+\mu(\Psi))=0, \tag{3.7}$$

and to prove that the second term tends to zero it is enough to show that

$$\lim_{n\to\infty}\sum_{i=1}^\infty ix_i\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)\geqslant n,\,\psi_i=0}\frac{x^\Psi}{\Psi!}(i+\mu(\Psi))=0. \tag{3.8}$$

Now (3.7) holds, because the sum on the l.h.s. of (3.7) can be estimated by

$$\sum_{i=n}^\infty i^2x_i\sum_{\Psi:\,|\Psi|\leqslant k}\frac{x^\Psi}{\Psi!}+\sum_{i=n}^\infty ix_i\sum_{\Psi:\,|\Psi|\leqslant k}\frac{x^\Psi}{\Psi!}\mu(\Psi)$$

and both terms here tends to zero as $n\to\infty$, since in each term the first multiplier tends to zero and the second is bounded (because $\mu_2(x)$ is bounded). Similarly (3.8) holds, because the sum on the l.h.s. of (3.8) can be estimated by

$$\sum_{i=1}^\infty i^2x_i\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)\geqslant n}\frac{x^\Psi}{\Psi!}+\sum_{i=1}^\infty ix_i\sum_{\Psi:\,|\Psi|\leqslant k,\,\mu(\Psi)\geqslant n}\frac{x^\Psi}{\Psi!}\mu(\Psi)$$

and again both terms here tend to zero as $n \to \infty$, since in each term the second multiplier tends to zero and the first is bounded.

Lemma 3.2. Let $x = \{x_1, x_2,...\}$ and $y = \{y_1, y_2,...\}$ have bounded second moments $\mu_2(x)$ and $\mu_2(y)$. Let $z_i = x_i - y_i$ and $\sigma_i = \text{sign}(x_i - y_i)$ (i.e., σ_i is 1, zero, or -1 respectively if $x_i - y_i$ is positive, zero, or negative). Then

$$\sum_{i=1}^{n} i\sigma_i(f_i^n(x) - f_i^n(y)) \leqslant \sigma \sum_{i=1}^{n} i\,|z_i|, \tag{3.9}$$

where f_i^n are defined by (2.4) and where the constant σ depends only on $c, k, \mu_2(x), \mu_2(y)$ and not on n.

Proof. By (2.4), the l.h.s. of (3.9) can be written as

$$\sum_{i=1}^{n} i\sigma_i \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant n} \frac{x^\Psi - y^\Psi}{\Psi!} \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_\Psi^\Phi(\phi_i - \psi_i),$$

or using (1.13) even more explicitly as

$$\begin{aligned}
&\sum_{l=1}^{k} \frac{1}{l!} \sum_{i_1,...,i_l:\, i_1+\cdots+i_l \leqslant n} (x_{i_1}\cdots x_{i_l} - y_{i_1}\cdots y_{i_l}) \\
&\quad\times \sum_{\Phi:\, \mu(\Phi) = i_1+\cdots+i_l} P^\Phi_{i_1e_1+\cdots+i_le_l}\left(\sum_{i=1}^{n} i\sigma_i\phi_i - i_1\sigma_{i_1} - \cdots - i_l\sigma_{i_l}\right) \\
&= \sum_{l=1}^{k} \frac{1}{l!} \sum_{i_1,...,i_l\,:\, i_1+\cdots+i_l \leqslant n} \sum_{m=1}^{l} x_{i_1}\cdots x_{i_{m-1}} z_{i_m} y_{i_{m+1}}\cdots y_{i_l} \\
&\quad\times \sum_{\Phi:\, \mu(\Phi) = i_1+\cdots+i_l} P^\Phi_{i_1e_1+\cdots+i_le_l}\left(\sum_{i=1}^{n} i\sigma_i\phi_i - i_1\sigma_{i_1} - \cdots - i_l\sigma_{i_l}\right).
\end{aligned} \tag{3.10}$$

Using the inequality

$$\sum_{i=1}^{n} i\sigma_i\phi_i \leqslant \sum_{i=1}^{n} i\phi_i = \mu(\Phi) = \mu(\Psi) = i_1 + \cdots + i_l,$$

and (2.8) we can estimate

$$\begin{aligned}
&\sum_{\Phi:\, \mu(\Phi) = i_1+\cdots+i_l} P^\Phi_{i_1e_1+\cdots+i_le_l}\left(\sum_{i=1}^{n} i\sigma_i\phi_i - i_1\sigma_{i_1} - \cdots - i_l\sigma_{i_l}\right) z_{i_m} \\
&\leqslant (i_1+\cdots+i_l)(i_1+\cdots+i_l - i_1\sigma_{i_1} - \cdots - i_l\sigma_{i_l})\, \sigma_{i_m}\, |z_{i_m}| \\
&\leqslant 2(i_1+\cdots+i_l)(i_1+\cdots+i_l - i_m)\, |z_{i_m}|.
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) we conclude that

$$\sum_{i=1}^{n} i\sigma_i(f_i^n(x)-f_i^n(y)) \leqslant A+B, \tag{3.12}$$

where

$$A=2\sum_{l=1}^{k}\frac{1}{l!}\sum_{i_1=1}^{n}\cdots\sum_{i_l=1}^{n}\sum_{m=1}^{l} x_{i_1}\cdots x_{i_{m-1}}\, y_{i_{m+1}}\cdots y_{i_l}\, |z_{i_m}|$$
$$\times i_m(i_1+\cdots+i_{m-1}+i_{m+1}+\cdots+i_l),$$
$$B=4\sum_{l=1}^{k}\frac{1}{l!}\sum_{i_1=1}^{n}\cdots\sum_{i_l=1}^{n}\sum_{m=1}^{l} x_{i_1}\cdots x_{i_{m-1}}\, y_{i_{m+1}}\cdots y_{i_l}\, |z_{i_m}|$$
$$\times [(i_1+\cdots+i_{m-1})^2+(i_{m+1}+\cdots+i_l)^2].$$

By (2.15)

$$A\leqslant 2\sum_{l=1}^{k}\frac{1}{(l-1)!}\sum_{i_1=1}^{n}\cdots\sum_{i_l=1}^{n}\sum_{m=1}^{l} x_{i_1}\cdots x_{i_{m-1}}\, y_{i_{m+1}}\cdots y_{i_l}\, |z_{i_m}|$$
$$\times i_m(i_1\cdots i_{m-1} i_{m+1}\cdots i_l)$$
$$\leqslant 2\sum_{i=1}^{n} i\,|z_i|\sum_{l=1}^{k}\frac{1}{(l-1)!}\sum_{m=1}^{l}(\mu(x))^{m-1}\,(\mu(y))^{l-m}$$
$$\leqslant 2\sum_{i=1}^{n} i\,|z_i|\sum_{l=1}^{k}\frac{1}{(l-1)!}(\mu(x)+\mu(y))^{l-1}\leqslant 2\sum_{i=1}^{n} i\,|z_i|\, e^{\mu(x)+\mu(y)}.$$

Similarly

$$B\leqslant 4\sum_{i=1}^{n}|z_i|\sum_{l=1}^{k}\frac{1}{l!}\sum_{m=1}^{l}(m-1)^2\,(l-m)^2\,\mu_2(x)^{m-1}\,\mu_2(y)^{l-m}$$
$$\leqslant 4\sum_{i=1}^{n}|z_i|\sum_{l=1}^{k} l^3\frac{1}{(l-1)!}(\mu_2(x)+\mu_2(y))^{l-1}$$
$$\leqslant 4k^3\sum_{i=1}^{n}|z_i|\, e^{\mu_2(x)+\mu_2(y)}.$$

These estimates together with (3.12) clearly imply (3.9).

Proof of Theorem 3.1. Denoting $\sigma_i(t)=\operatorname{sign}(x_i(t)-y_i(t))$ we can write

$$|x_i(t)-y_i(t)|=|\xi_i-\eta_i|+\int_0^t\sigma_i(\tau)(f_i(x(\tau))-f_i(y(\tau)))\,d\tau.$$

Consequently

$$\mu(\mathcal{P}_n(x(t)) - \mathcal{P}_n(y(t)))$$

$$= \sum_{i=1}^{n} i\, |x_i(t) - y_i(t)|$$

$$\leqslant \sum_{i=1}^{n} i\, |\xi_i - \eta_i| + \sum_{i=1}^{n} \int_0^t \sigma_i(\tau)\, i(f_i^n(x(\tau)) - f_i^n(y(\tau)))\, d\tau$$

$$+ \sum_{i=1}^{n} \int_0^t i\, |f_i(x(\tau)) - f_i^n(x(\tau))|\, d\tau + \sum_{i=1}^{n} \int_0^t i\, |f_i(y(\tau)) - f_i^n(y(\tau))|\, d\tau.$$

By Lemmas 3.1 and 3.2 this implies that

$$\sum_{i=1}^{n} i\, |x_i(t) - y_i(t)| \leqslant \sum_{i=1}^{n} i\, |\xi_i - \eta_i| + \sigma \int_0^t \sum_{i=1}^{n} i\, |x_i(\tau) - y_i(\tau)|\, d\tau + o(1),$$

where $o(1)$ tends to zero as $n \to \infty$. Passing to the limit as $n \to \infty$ we obtain

$$\mu(x(t) - y(t)) \leqslant \mu(\xi - \eta) + \sigma \int_0^t \mu(x(\tau) - y(\tau))\, d\tau,$$

which implies (3.1) by Gronwall's lemma. Theorem 3.1 is proved.

In the following theorem we collect some more or less direct consequences of Theorems 2.2 and 3.1 on the properties of solutions to (2.1).

Theorem 3.2. Suppose (2.8) holds and $x_0 \in \mathcal{M}_c^2$ with some $c > 0$. Then

(i) there exists a unique weak solution $x(t) = X(t, x_0)$ of (2.1) with the initial condition x_0 and such that $\mu_2(x(t))$ is uniformly bounded on $t \in [0, T]$ for any positive T; this solution can be equivalently characterized as a unique weak solution of (2.1) such that it is a limit in c_∞ of a subsequence of the sequence of solutions $x^n(t)$ of the equations $\dot{x} = f^n(x)$ with initial conditions $x_0^n = \mathcal{P}_n x_0$; moreover, this weak solution is in fact a c_p-solution for any $p > 1$ and $x(t) \in \mathcal{M}_c^2$ for all t;

(ii) the solution $X(t, x_0)$ is the limit in the topology of $\mathcal{M}$ (i.e., in μ-norm) of the whole sequences (not just its subsequence) of the finite dimensional approximations $x^n(t)$ described above;

(iii) the solution $X(t, x_0)$ is a continuous function of two variables t and x_0 for $x_0 \in \bigcup_{c \geqslant 0} \mathcal{M}_c^2$, where x_0 and $X(t, x_0)$ are considered in the topology of $\mathcal{M}$;

(iv) if (2.16) or (2.17) hold, then the function $\mu_2(x(t))$ is locally Lipschitz continuous function of t which is therefore absolutely continuous and almost everywhere differentiable; the estimate

$$-a\mu_2(X(t,x_0))-b\leqslant\frac{d}{dt}\mu_2(X(t,x_0))\leqslant a\mu_2(X(t,x_0))+b \tag{3.13}$$

holds for the derivative with some constants a, b depending on c, k and constants C and ω in (2.8), (2.16), or (2.17); this implies

$$\mu_2(X(t,x_0))\geqslant e^{-at}\mu_2(x_0)-b/a. \tag{3.14}$$

Remarks. (1) For a discussion of conditions (2.16) and (2.17) see the remark after Lemma 2.4. (2) From (iv) it follows, as one can expect that in processes of pure coagulation when $\mu_2(\Phi)\geqslant\mu_2(\Psi)$ whenever $P_\Psi^\Phi\neq 0$, the function $\mu_2(x(t))$ does not decrease on the solution $x(t)=X(t,x_0)$.

Proof. (i) is immediate from Theorems 2.2 and 3.1. (ii) As follows from our proof of Theorem 2.2, from the sequence $x^n(t)$ and similarly from any its subsequence, one can choose a subsequence converging in c_∞ to a c_∞-solution of (2.1). As such limiting solution $X(t,x_0)$ of (2.1) is unique by (i), we conclude that the whole sequence $x^n(t)$ converges in c_∞ to $X(t,x_0)$. But as all $\mu_2(x^n(t))$ are locally (in t) uniformly (in n) bounded, the c_∞-convergence implies the convergence in $\mathcal{M}$. (iii) As $X(t,x_0)$ is a c_∞-solution of (2.1), it depends continuously on t in c_∞-topology. But again as above, as all $\mu_2(x^n(t))$ are locally (in t) uniformly (in n) bounded, the continuity in c_∞-norm implies the continuity in μ-norm. On the other hand, by Theorem 3.1, $X(t,x_0)$ is continuous in x_0 locally uniform in t, which implies the required joint continuity of $X(t,x_0)$ as a function of two variables. (iv) $\mu_2(x^n(t))$ is differentiable for finite-dimensional approximations $x^n(t)$ of the solution $X(t,x_0)$. Of course, one should be careful with differentiability when passing from $x^n(t)$ to $X(t,x_0)$. We proceed as follows. From (2.13)

$$\mu_2(x^n(t+\tau))\leqslant\mu_2(x^n(t))+\tau(a\mu_2(x^n(t))+\beta+\epsilon)$$

for an arbitrary ϵ and for τ small enough. As $x^n(t)$ converges to $X(t,x_0)$ in c_∞-norm, it implies

$$\mu_2(\mathcal{P}_mX(t+\tau,x_0))\leqslant\mu_2(\mathcal{P}_mX(t,x_0))+\tau(a\mu_2(\mathcal{P}_mX(t,x_0))+\beta+\epsilon)$$

for any m. Passing to the limit as $m \to \infty$ we get the same inequality for $X(t, x_0)$ instead of $\mathscr{P}_m X(t, x_0)$. As ϵ is arbitrary we then conclude that

$$\limsup_{\tau \to 0} \frac{\mu_2(X(t+\tau, x_0)) - \mu_2(X(t, x_0))}{\tau} \leqslant a\mu_2(X(t, x_0)) + \beta. \qquad (3.15)$$

The same arguments lead from (2.16) or (2.17) and Lemma 2.4 to the estimate

$$\liminf_{\tau \to 0} \frac{\mu_2(X(t+\tau, x_0)) - \mu_2(X(t, x_0))}{\tau} \geqslant -\tilde{a}\mu_2(X(t, x_0)) - \tilde{\beta}. \qquad (3.16)$$

Together (3.15) and (3.16) imply the Lipschitz continuity of $\mu(X(t, x_0))$ and estimates (3.13) Clearly (3.14) follows from (3.13). Theorem 3.2 is proved.

To conclude this section, we discuss some consequences of Theorem 3.2 to the analysis of the semigroup generated by operator (1.6). If X is a topological space that is a union $X = \bigcup_{n=1}^{\infty} K_n$ of an increasing sequence of compact subsets K_n, let us denote by $C_\infty(X)$ the Banach space of bounded continuous functions f on X (with the usual sup-norm) vanishing at infinity, i.e., such that for an arbitrary ϵ, there exists n such that $|f(x)| \leqslant \epsilon$ for $x \notin K_n$. A semigroup of positivity preserving contractions T_t, $t \geqslant 0$, on $C_\infty(X)$ is called a *Feller semigroup* if it is strongly continuous in t, i.e., $\|T_t f - f\| \to 0$ as $t \to 0$ for any $f \in C_\infty$. This definition is slightly more general than the usual one where the topological space X is considered to be locally compact (for a wide discussion of the theory of Feller semigroups we refer to ref. 17).

In the following we shall consider the set $\mathscr{M}_c^2$ with the topology induced from $\mathscr{M}$, i.e., as a metric space with the metric $\mu(x-y)$. Clearly, $\mathscr{M}_c^2$ is a complete metric space that is the union $\bigcup_{n=1}^{\infty} K_n$ of the compact sets $K_n = \{x \in \mathscr{M}_c^2 : \mu_2(x) \leqslant n\}$.

Theorem 3.3. (i) If (2.8) holds, the family of operators T_t on $B(\mathscr{M}_c^2)$ defined as $T_t f(x) = f(X(t, x))$ is a semigroup of positivity preserving contractions on $B(\mathscr{M}_c^2)$, which preserves the subspace $C_b(\mathscr{M}_c^2)$ of continuous functions. Moreover, for any $f \in C_b(\mathscr{M}_c^2)$, $T_t f(x)$ tends to $f(x)$ as $t \to 0$ uniformly for x from any K_n. (ii) If additionally (2.16) or (2.17) hold, then T_t is a Feller semigroup on $C_\infty(\mathscr{M}_c^2)$.

Proof. (i) It is immediate from Theorem 3.2(i) and (iii). (ii) To deduce (ii) from (i) one only needs to show that the space $C_\infty(\mathscr{M}_c^2)$ is preserved by T_t. But this follows from Theorem 3.2 (iv), namely from estimate (3.14).

4. CONVERGENCE OF STOCHASTIC APPROXIMATIONS

Unlike previous sections we shall use here probabilistic tools. Doing this, we shall denote by the capital letters E and P the expectation and respectively the probability defined by a process under consideration. For a metric space M we shall use the standard notation $D_M[0, \infty)$ to denote the Skorohod space of càdlàg paths in M.

Let $X_h^{Nh}(t)$ be the Markov chain in $h\mathbf{Z}_{+,\mathrm{fin}}^{\infty}$ (with càdlàg sample paths) defined by the generator (1.4) and an initial condition Nh. This Markov chain is well defined, because it has only a finite number of states. This section is devoted to a proof of the convergence in distribution of the Markov chain $X_h^{Nh}(t)$ to the deterministic process described by equations (1.7). This result will be obtained as a consequence of the following theorem that gives an alternative (probabilistic) proof of the main existence results for solutions to (1.7) obtained in Section 2.

Theorem 4.1. Let (2.2) (respectively (2.8)) hold and let the family $Nh = N(h)\, h$ of points from $h\mathbf{Z}_{+,\mathrm{fin}}^{\infty}$ have a uniformly bounded mass, i.e., $\mu(Nh) \leqslant c$ for all h and some finite c, (respectively a uniformly bounded second mass moment, i.e., $\mu_2(Nh) \leqslant d$ for all h and some finite d), and moreover, Nh converges in c_∞-norm as $h \to 0$ to a point $x \in M_{\leqslant c}$. Then there exists a subsequence of the family $X_h^{Nh}(t)$ that converges as a family of processes with sample paths in $D_{c_\infty}[0, \infty)$ (respectively, in $D_{\mathcal{M}}[0, \infty)$), to a deterministic process $X^x(t)$ with continuous trajectories that are weak solutions of (2.1).

Remark. For the case of standard Smoluchovski's equations (1.10), the analogous result was proved in ref. 18 for the discrete mass models and then generalized in ref. 29 for the continuous mass model (without fragmentation, however).

Proof. By Dynkin's formula,

$$M_g(t) = g(X_h^{Nh}(t)) - g(Nh) - \int_0^t G_k^h g(X_h^{Nh}(\tau))\, d\tau \tag{4.1}$$

is a martingale for any function g on the (finite) state space of the Markov chain X_h^{Nh}. The idea of the proof is to show the tightness of the family of processes X_h^{Nh}, then to choose a subsequence converging to some process X^x, and then to pass to the limit as $h \to 0$ in (4.1) with the test function $g(x) = g_j(x) = x_j$ to obtain

$$0 = (X^x(t))_j - x_j - \int_0^t \Lambda_k\, g_j(X^x(\tau))\, d\tau, \tag{4.2}$$

which would mean precisely that $X^x(t)$ are weak solutions of (2.1). The formal implementation of this programme will be divided in the following four steps.

Step 1. If (2.2) (respectively (2.8)) holds, then for the family X_h^{Nh} the compact containment condition holds, i.e., for arbitrary $\eta > 0$, $T > 0$ there exists a compact subset $\Gamma_{\eta,T} \subset c_\infty$ (respectively $\Gamma_{\eta,T} \subset \mathscr{M}$) for which

$$\inf_h P(X_h^{Nh}(t) \in \Gamma_{\eta,T} \text{ for } 0 \leqslant t \leqslant T) \geqslant 1-\eta. \tag{4.3}$$

If (2.2) holds, (4.3) is obvious, because as masses of $X_h^{Nh}(t)$ are uniformly bounded, they all lie in a compact set of c_∞. To prove (4.3) for the case of (2.8) with $\Gamma_{\eta,T}$ being a compact subset of $\mathscr{M}$, we shall follow the line of arguments from Lemma 2.3 to show that with the probability arbitrary close to one, the second mass moment of the family $X_h^{Nh}(t)$ is uniformly bounded for $t \leqslant T$ with any $T > 0$, and hence $X_h^{Nh}(t)$ lie in a compact subset. Using the martingale property of the process (4.1) with the test function $g(x) = \mu_2(x)$ yields

$$E\mu_2(X_h^{Nh}(t)) = \mu_2(Nh) + \int_0^t EG_k^h \mu_2(X_h^{Nh}(\tau))\, d\tau. \tag{4.4}$$

As

$$\begin{aligned} G_k^h \mu_2(Nh) &= \frac{1}{h} \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant \mu(Nh)} C_\Psi^h(Nh) \\ &\quad \times \sum_{\Phi:\, \mu(\Phi) = \mu(\Psi)} P_\Psi^\Phi(\mu_2(Nh - \Psi h + \Phi h) - \mu_2(Nh)) \\ &\leqslant \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant \mu(Nh)} \frac{(Nh)^\Psi}{\Psi!} \sum_j j^2(\phi_j - \psi_j), \end{aligned}$$

we use (as in the proof of Lemma 2.3) (2.8) and (2.10) to get

$$G_k^h \mu_2(Nh) \leqslant C \sum_{\Psi:\, |\Psi| \leqslant k,\, \mu(\Psi) \leqslant n} \frac{x^\Psi}{\Psi!} \mu(\Psi)[(\mu(\Psi))^2 - \mu_2(\Psi)]$$

and consequently (see again the proof of Lemma 2.3)

$$G_k^h \mu_2(Nh) \leqslant a\mu_2(Nh) + \beta.$$

Hence from (4.4) and Gronwall's lemma one obtains

$$E\mu_2(X_h^{Nh}(t)) \leqslant e^{at}(\mu_2(Nh) + \beta/a).$$

Since

$$\mu_2(X_h^{Nh}(t)) \leqslant |M(t)| + \left| \int_0^t G_k^h \mu_2(X_h^{Nh}(\tau))\, d\tau \right|,$$

one gets using the maximal inequality for the submartingale on the r.h.s. of this inequality that

$$rP(\sup_{t \leqslant T} |\mu_2(X_h^{Nh}(t))| \geqslant r) \leqslant C(T)(\mu_2(Nh)+1)$$

with some constant $C(T)$. This implies (4.3).

Step 2. Let $[M_g](t)$ denote the quadratic variation of the martingale (4.1). If $g(x)$ is finite-dimensional, i.e., $g(x) = g(\mathscr{P}_n(x))$ for some n and all x, and moreover, g is continuously differentiable with the uniformly bounded derivative, then

$$E([M_g(t)] - [M_g(s)]) \leqslant \sigma h(t-s) \tag{4.5}$$

with some constant σ uniformly for all $0 \leqslant s \leqslant t \leqslant T$ and an arbitrary T.

As the integral on the r.h.s. of (4.1) is a continuous process with a bounded variation, we conclude that

$$[M_g(t)] = [g(X_h^{Nh}(t))] = \sum_{s \leqslant t} (\Delta g(X_h^{Nh}(s)))^2,$$

where $\Delta Z(s) = Z(s) - Z(s_-)$ denotes the jump of a process $Z(s)$. As (2.8) implies (2.3) with $\alpha = 1$, it follows from (1.5) for both cases (2.2) and (2.8) that all possible jumps of $g(X_h^{Nh}(t))$ are uniformly bounded by $2kh$ and that the expectation of the number of jumps on the interval $[s, t] \subset [0, T]$ does not exceed

$$\frac{t}{h} \sum_{\Psi} \frac{1}{\Psi!} \sup_{r \in [s,t]} (X_h^{Nh}(r))^{\Psi} \sum_{\Phi:\, \mu(\Psi) = \mu(\Phi)} P_{\Psi}^{\Phi}$$
$$\leqslant \frac{tC}{h} \sum_{\Psi} \frac{1}{\Psi!} \sup_{r \in [s,t]} (X_h^{Nh}(r))^{\Psi} \prod_{j=1}^{\infty} j^{\psi_j} \leqslant \frac{tC}{h} \sum_{l=1}^{k} \frac{\mu(Nh)^l}{l!}$$

with some constant C. Hence

$$E([M_g(t)] - [M_g(s)]) \leqslant 4k^2 h(t-s)\, C \sum_{l=1}^{k} \frac{\mu(Nh)^l}{l!},$$

which implies (4.5).

Step 3. If (2.2) (respectively (2.8)) holds, the family of processes $X_h^{Nh}(t)$ is tight as a family of processes with sample paths in $D_{c_\infty}[0,\infty)$ (respectively, in $D_{\mathcal{M}}[0,\infty)$).

As the compact containment condition (4.3) holds, by the well known criterion (see, e.g., Theorem 9.1 from Chapter 3 in ref. 16), to prove tightness one must show that the family of real-valued processes $g(X_h^{Nh}(t))$ is relatively compact (as a family of processes with sample paths in $D_{\mathbf{R}}[0,\infty)$) for any finite-dimensional g from Step 2. To this end, by standard tightness criteria for real valued processes (see, e.g., Corollary 7.4 from Chapter 3 of ref. 16 or the Aldous–Rebolledo criterion in ref. 12) one needs to estimate the oscillations of $X_h^{Nh}(t)$. As the integral part in (4.1) is continuous with finite variation, to estimate its oscillations one only needs to estimate the oscillations of the quadratic variation $[M_g](t)$. But for $[M_g](t)$ all required estimates follow from (4.5).

Step 4. End of the proof of Theorem 4.1.

It remains to show that the limit $X^x(t)$ of a converging subsequence $X_h^{Nh}(t)$, $h \to 0$ and belong to a countable set, is a weak solution of (2.1). As we mentioned above, we are going to use (4.1) with the test function $g(x) = g_j(x) = x_j$ to obtain (4.2). From Step 2 it follows that the martingale on the l.h.s. of (4.1) with this g tends to zero almost surely. Clearly, the first two terms on the r.h.s. of (4.1) with this g will tend to the first two terms on the r.h.s. of (4.2). So, we need to show that the integral $\int_0^t G_k^h g_j(X_h^{Nh}(t))\, dt$ tends to the integral on r.h.s. of (4.2). As $|(G_k^h g_j - \Lambda_k g_j)(x)|$ tends to zero as $h \to 0$ uniformly for all x with a uniformly bounded mass, we need only to show that

$$|\Lambda_k g_j(X_h^{Nh}(t)) - \Lambda_k g_j(X^x(t))| \to 0,$$

or more explicitly that for any j

$$\int_0^t (f_j(X_h^{Nh}(s)) - f_j(X^x(s)))\, ds \to 0. \tag{4.6}$$

But from a weak convergence it follows (see, e.g., ref. 16) that $X_h^{Nh}(s)$ converges to $X^x(s)$ for all $s \in [0,t]$ apart from some countable subset. As the function f is uniformly continuous on $\mathcal{M}_{\leqslant c}$ for any positive c (because, as shown in our proof of Lemma 2.2, it is a uniform limit of uniformly continuous functions f^n), it follows that the difference under the integral in (4.6) is uniformly bounded and tends to zero for all s apart from some countable subset. This implies (4.6) and completes the proof of Theorem 4.1.

As an immediate consequence of Theorem 4.1(ii), Theorem 3.2(i), (ii), and Theorem 3.3(ii) we obtain now the following main result of this paper.

Theorem 4.2. Let (2.8) hold and let the family $Nh = N(h)\,h$ of points from $h\mathbf{Z}^{\infty}_{+,\,\mathrm{fin}}$ have a uniformly bounded second mass moment, i.e., $\mu_2(Nh) \leqslant d$ for all h and some finite d, and moreover, Nh converges in c_∞-norm as $h \to 0$ to a point $x \in M^2_c$ with some finite c. Then the family $X^{Nh}_h(t)$ with paths in $D_{\mathscr{M}}[0, \infty)$ converges to a deterministic process $X^x(t)$ with continuous trajectories that are (mass-conserving) c_p-solutions of (2.1) with any $p > 1$. If additionally (2.16) or (2.17) hold, then the Feller semigroup of the Markov chain $X^{Nh}_h(t)$ tends to the Feller semigroup on $C_\infty(\mathscr{M}^2_c)$ defined by the solutions of (1.7).

5. DIFFUSION APPROXIMATION FOR MASS EXCHANGE PROCESSES

As was shown in this paper, the uniform scaling (1.4) of the Markov mass exchange process defined by (1.3) leads to the deterministic measure-valued limit described by kinetic equations (1.7). In the light of the recent increase of interest to stochastic measure-valued limits (see, e.g., ref. 12; in case of branching processes such limits are called superprocesses, see ref. 15 for a review), one can naturally ask about possible stochastic measure-valued limits of (1.3) under an appropriate scaling. For general k-nary interacting particle systems with a finite number of types of the particles such limits were studied in ref. 20. It turns out that the conservation of mass property poses certain restrictions to the existence of nondeterministic limits (diffusion approximation requires some symmetry of the process), and such limits seem to be not available for generators (1.3) with $k \leqslant 2$, i.e., for binary interactions, as well as for processes with pure coagulation or fragmentation. Nevertheless, as we are going to show, for some process of type (1.3), the natural diffusion approximation can be constructed. We shall not discuss here this approximation in the most general situation, but rather for the simplest concrete model. This model does not look very realistic physically, as it assumes some sort of pattern behavior of particles. Possibly, it can be better interpreted in the biological context of ref. 31.

Consider a process with generator (1.3) where $P^{\Phi}_{\Psi} \neq 0$ only for Ψ consisting of three particles such that the sum of masses of two of them equals the mass of the third, i.e., for $\Psi = e_i + e_j + e_{i+j}$. Next, suppose that as the result of a collision (or interaction) of three particles of mass i, j, $i+j$ either the particle of mass $i+j$ will fragment into two pieces of mass i and j or the particles with masses i and j will coagulate into a single particle. Under these assumptions the generator (1.3) will take the form

$$Gf(N) = \sum_{i,j} n_i(n_j - \delta_i^j)\, n_{i+j}\, [P_{ij}^f(f(N - e_{i+j} + e_i + e_j) - f(N)) \\ + P_{ij}^c(f(N + e_{i+j} - e_i - e_j) - f(N))].$$

Assuming further that

$$P_{ij}^f = \frac{1}{h} a_{ij} + p_{ij}^f, \qquad P_{ij}^c = \frac{1}{h} a_{ij} + p_{ij}^c$$

we get the corresponding scaled operator (1.5) in the form

$$G^h f(x) = \sum_{i,j} x_i(x_j - h\delta_i^j)\, x_{i+j} \left[\left(\frac{a_{ij}}{h^2} + \frac{p_{ij}^c}{h}\right)(f(Nh + he_{i+j} - he_i - he_j) - f(Nh)) \right. \\ \left. + \left(\frac{a_{ij}}{h^2} + \frac{p_{ij}^f}{h}\right)(f(Nh - he_{i+j} + he_i + he_j) - f(Nh))\right]. \tag{5.1}$$

Clearly, as $h \to 0$, this operator tends formally to

$$\Lambda f(x) = \sum_{i,j} x_i x_j x_{i+j} \left[(p_{ij}^f - p_{ij}^c)\left(\frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_{i+j}}\right) \right. \\ \left. + a_{ij}\left(\frac{\partial^2 f}{\partial x_i^2} + \frac{\partial^2 f}{\partial x_j^2} + \frac{\partial^2 f}{\partial x_{i+j}^2} + 2\frac{\partial^2 f}{\partial x_i\, \partial x_j} - 2\frac{\partial^2 f}{\partial x_i\, \partial x_{i+j}} - 2\frac{\partial^2 f}{\partial x_j\, \partial x_{i+j}}\right)\right]. \tag{5.2}$$

Let us give a rigorous result on the convergence of the corresponding stochastic processes for initial conditions $x \in \mathbf{R}^\infty_{+,\text{fin}}$, which is a consequence of the theory developed in ref. 20. Let A_n denote an operator on smooth functions on $\mathbf{R}^n$ with a compact support defined by formula (5.2) but with the sum over all i, j replaced by the sum over i, j such that $i + j \leqslant n$. Let $X_h^{Nh}(t)$ be the Markov chain in $h\mathbf{Z}^\infty_{+,\text{fin}}$ defined by the generator (5.1) and the initial condition Nh. Notice that due to a special structure of (5.1), $X_h^{Nh}(t)$ stays in $h\mathbf{Z}^n_+ \subset \mathbf{R}^n_+$ all times whenever $N \in \mathbf{Z}^n_+$. The following result is a direct consequence of Theorem 3 from ref. 20, which is in its turn a consequence of the theory developed in ref. 23.

Theorem 5.1. If $N \in \mathbf{Z}^n_+$, the Markov process $X_h^{Nh}(t) \in h\mathbf{Z}^n_+ \subset h\mathbf{Z}^\infty_{+,\text{fin}}$ converges in the sense of distributions to the (uniquely defined) conservative diffusion process on $\mathbf{R}^n_+$ with the generator A_n.

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